Analytical Solutions for Film Thinning Dynamics in Bubble Coalescence

Richard G. Rice
Dept. of Chemical Engineering, Louisiana State University, Baton Rouge, LA 70803

DOI 10.1002/aic.10742
Published online November 28, 2005 in Wiley InterScience (www.interscience.wiley.com).

Introduction

Unsolved problems on the theory of coalescence of bubbles often affect recent attempts at large scale modeling using population balances and CFD on bubble column reactors. Thus, recent work by Chen et al.[1] and Laari and Turunen[2] used an older approximate film thinning formula for calculating coalescence efficiency. To correct this, this work produces exact solutions for film thinning, which include either classical Hamaker force used by Marrucci,[3] or the retarded Van der Waals force suggested by Prince and Blanch.[4]

Theory

When bubbles collide, a small amount of liquid is entrapped between them, forming a small circular lens or film of radius R and thickness h. The forces causing the film or lens to grow thinner in pure systems arise from capillary pressure, augmented by compression from a close range Hamaker force which accounts for the mutual attraction of water molecules on opposite sides of the liquid film. For equal size bubbles, Oolman and Blanch[5] derived the thinning formula

$$-dh/dt = \left\{\frac{8}{R^2\rho_l}\left[h^2\left(\frac{2\sigma}{r_b}+\frac{A}{6\pi h^3}\right)\right]\right\}^{1/2} \tag{1}$$

Prince and Blanch[6] solved this equation numerically and concluded the added Hamaker force had only slight influence on coalescence time. They wrote that the added term was cumbersome in the numerical calculations. As we show presently, a numerical solution is not necessary and, in fact, an easy-to-use exact solution is revealed in this work. In a followup article, Prince and Blanch[4] suggest replacing the Hamaker inverse cubic law with a retarded Van der Waals force, so Eq. 1 was modified to read

$$-dh/dt = \left\{\frac{8}{R^2\rho_l}\left[h^2\left(\frac{2\sigma}{r_b}+\frac{B}{h^4}\right)\right]\right\}^{1/2} \tag{2}$$

The term B is the retarded Van der Waals coefficient, $B = 1.5 \times 10^{-19}$ erg-cm, compared to the Hamaker constant, $A = 2.5 \times 10^{-13}$ ergs.

Both of the thinning equations given above can be easily separable and have well known integrals, after a slight change in variables. Starting with Eq. 1, remove $1/\sqrt{h}$ from the right-hand side (RHS) and combine with the differential on the LHS as follows

$$-\sqrt{h}\frac{dh}{dt} = -\frac{2}{3}\frac{d(h^{3/2})}{dt} = 4\left(\frac{\sigma}{R^2\rho_l r_b}\right)^{1/2}(h^3+a^2)^{1/2} \tag{3}$$

where $a^2 = A/6\pi(r_b/2\sigma)$

Next, make the substitution $x = h^{3/2}$, and $x^2 = h^3$ and use the widely tabulated integral

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln(x+\sqrt{x^2+a^2}) \tag{4}$$

Hence, for the initial condition $h(0) = h_0$ the result is

$$\ln\left(\frac{h^{3/2}+\sqrt{h^3+a^2}}{h_0^{3/2}+\sqrt{h_0^3+a^2}}\right) = -6\left(\frac{\sigma}{R^2\rho_l r_b}\right)^{1/2} t \tag{5}$$

We can make these results dimensionless by defining $\xi = h/h_0$:

$$\ln\left(\frac{\xi^{3/2}+\sqrt{\xi^3+N_H}}{1+\sqrt{1+N_H}}\right) = -6\left(\frac{t}{\tau}\right) \tag{6}$$

where

R. G. Rice's e-mail address is dickrice@wwgap.net.

Table 1. Thinning Times for Air-Water System when $r_b = R = 2$ mm

Model	Time Constants $\left(\frac{R^3\rho_l}{\sigma}\right)^{1/2} = 0.0105\ s$	Time to Reach $h_f/h_0 = 10^{-4}$	Time to Reach $h_f/h_0 = 0$
Hamaker, Eq. 6 $N_H = 1.842 \times 10^{-11}$	$\left(\frac{\tau}{6}\right) = 0.00175\ s$	0.022440 s	0.022845 s
Van der Waal, Eq. 7 $N_V = 2.083 \times 10^{-14}$	$\left(\frac{\tau}{8}\right) = 0.00131\ s$	0.021455 s	0.021546 s
Approximate, Eq. 8	$\left(\frac{\tau}{4}\right) = 0.00263\ s$	0.024228 s	∞

$$N_H = \frac{a^2}{h_0^3} = \frac{1}{12\pi}\left(\frac{Ar_b}{\sigma h_0^3}\right)$$

$$\tau = \left(\frac{R^2\rho_l r_b}{\sigma}\right)^{1/2}$$

In the same way the Van der Waals model in Eq. 2 can be solved, except this time remove 1/h from the RHS and let $x = h^2$ and $x^2 = h^4$, then follow the previous procedure using $\int dx/(\sqrt{x^2 + a^2})$ and finally obtain

$$\ln\left(\frac{\xi^2 + \sqrt{\xi^4 + N_V}}{1 + \sqrt{1 + N_V}}\right) = -8\left(\frac{t}{\tau}\right) \tag{7}$$

where

$$N_V = \frac{1}{2}\left(\frac{Br_b}{\sigma h_0^4}\right)$$

Sample Calculations

Prince and Blanch[6] suggest that the initial film thickness, h_0, in air-water systems is around 10^{-4}m, and the final thickness, h_f, before rupture was around 10^{-8}m so that the dimensionless ratio at rupture was 10^{-4}. Moreover, the lens radius was taken to be the same as the bubble radius as an approximation, which simplifies the calculation to find the final time to coalescence t_f. The traditional approximation ignores both the Hamaker and Van der Waals contributions, and yields

$$\frac{t_f}{\tau} = \frac{1}{4}\ln\left(\frac{h_0}{h_f}\right) \tag{8}$$

Table 1 illustrates the different values of thinning times. The exact model allows computation of the actual rupture time when the film disappears, $h_f/h_0 = 0$, which is not possible to do with the approximate results in Eq. 8. In fact, the reason workers[6] need estimates of h_f to find the coalescence time is quite simple: taking the thickness to be zero leads to an infinite time for coalescence to occur.

Conclusions

Now that exact solutions are at hand, researchers need not ponder and guess at the final film thickness before rupture occurs, since this effort provides the means to determine the final time for coalescence. Table 1 shows for the air-water system that the Hamaker and Van der Waals models gave coalescence times about 10% less than the approximate model. These comparisons assumed $r_b \approx R$, and $r_b = 2$mm. Comparisons would change for other model conditions.

Notation

A = Hamaker constant, erg
B = retarded Van der Waals constant, erg-cm
h = film thickness between coalescing bubbles, m
h_o = initial film thickness, m
h_f = final film thickness, m
R = radius of lens joining bubbles, m
r_b = radius of bubble, m
t = time, s
t_f = final time, s

Greek

ξ = h/h_0, dimensionless thickness
ρ_l = liquid density, kg/m^3
σ = surface tension, kg/s^2
τ = $(R^2\rho_l r_b/\sigma)^{1/2}$, time constant, s

Literature Cited

1. Chen P, Dudukovic M P, Sanyal J. Three-Dimensional Simulation of bubble column flows with bubble coalescence and breakup. *AIChE J*. 2005;51:696-712.
2. Laari A, Turunen I. Experimental determination of bubble coalescence and break-up rates in a bubble column reactor. *Can J Chem Eng*. 2003;81:395 - 401.
3. Marrucci G. A theory of coalescence. *Chem Eng Sci*. 1969;24:975.
4. Prince M J, Blanch H W. Transition electrolytes concentrations for bubble coalescence. *AIChE J.* 1990;36:1425 - 1429.
5. Oolman T, Blanch H W. Bubble coalescence in stagnant liquids. *Chem Eng Comm*. 1986;43:237.
6. Prince M J, Blanch H W. Bubble coalescence and break-up in air-sparged bubble columns. *AIChE J.* 1990;36:1485 - 1499.

Manuscript received Aug. 25, 2005, and revision received Oct. 17, 2005.